Instability of very long waves in a zonal flow

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ABSTRACT

The eigenvalue problem governing baroclinic disturbances, relative to a zonal flow, of wavelengths comparable with the circumference of the Earth is formulated on the basis of: (*a*) adiabatic, frictionless motion of a perfect gas in a uniform gravitational field over a spherical Earth and (*b*) the restriction $\mathrm{Ro} < \mathrm{Ri}^{-\frac{1}{2}} < 1$, where Ro and Ri are appropriate Rossby and Richardson numbers. Attention is focused on unstable disturbances, and several theorems governing the existence of such disturbances are deduced from an earlier and more general study of baroclinic stability. Explicit results for the complex wave speed of an unstable disturbance are given in the limits $\beta \to 0$ and $\beta \to \infty$, where $\beta = \mathrm{Ro}\,\mathrm{Ri}\cot\varphi$ (φ = latitude). It is established that one and only one unstable mode exists for a wide class of monotonically increasing wind profiles.

1. Introduction

We consider here the stability of baroclinic disturbances, relative to a zonal flow, having wavelengths comparable with the circumference of the Earth. Let U_1 be a characteristic wind speed for the zonal flow, $f = 2\Omega \sin\varphi$ the usual Coriolis parameter, a the radius of the Earth, H the scale height of the atmosphere, g the acceleration of gravity, S the entropy, c_p the specific heat at constant pressure,

$$\varkappa \sim (H/c_p)(\partial S/\partial r) = Hs \tag{1.1}$$

a dimensionless measure of the stability (s is the conventional *stability*, namely the vertical gradient of the logarithm of the potential temperature),

$$\mathrm{Ro} = U_1/fa \tag{1.2}$$

an appropriate Rossby number, and

$$\mathrm{Ri} = \varkappa g H/U_1^2 \tag{1.3}$$

an appropriate Richardson number. We shall develop approximations to the equations of motion on the basis of the restrictions

$$\mathrm{Ro} < \mathrm{Ri}^{-\frac{1}{2}} < 1, \quad \varkappa < 1. \tag{1.4a, b}$$

The special character of very long waves, subject to the restrictions (1.4a, b), was discussed originally by BURGER (1958) and subsequently by WELANDER (1961), WIIN-NIELSEN (1961), and PHILLIPS (1963). (Burger and Wiin-Nielsen designated such waves as "planetary waves"; however, this same term is sometimes used to describe any wave that owes its existence to the rotation of the Earth.) Welander considered only neutral waves and gave explicit solutions for such waves for a particular model in which both wind speed and density are linear functions of the static pressure. Wiin-Nielsen invoked further simplifications in the mathematical model but did not restrict his analysis to neutral waves; he concluded that "unstable waves will appear for sufficiently large wind shear".

We shall base the following analysis on a mathematical model that is basically similar to the model developed by Welander. Given profiles of wind speed and temperature, Welander's model yields an eigenvalue problem for the determination of the wave speed $c = c(\beta)$, where

$$\beta = -\mathrm{Ri}\,\frac{\partial\,\mathrm{Ro}}{\partial\varphi} = \mathrm{Ro}\ \mathrm{Ri}\ \cot\ \varphi = \frac{\varkappa g H\ \cot\ \varphi}{2\Omega U_1 a\ \sin\ \varphi}. \tag{1.5}$$

Complex values of $c = c_r + ic_i$ necessarily occur in conjugate pairs, and $c_i > 0$ implies instability. This model is essentially a limiting case of the more general model for baroclinic instability, in

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which the restrictions $\mathrm{Ro} = kU_1/f < 1$ and $\mathrm{Ri} \gg 1$, but not $ka \sim 1$ as implied by (1.2), are imposed. A critical study of this more general model has been given elsewhere (MILES, 1964a), and the majority of the results given below follow more or less directly from that study.

2. Equations of motion

We consider adiabatic, frictionless motion of a perfect gas in a uniform gravitational field over a spherical Earth. Led by the preceding considerations of scale and the fact that the entropy of fluid particles is conserved (by virtue of the hypothesis of adiabatic motion), we choose as independent variables the longitude and latitude, λ and φ, and the dimensionless entropy and time

$$\sigma = S/\varkappa c_p, \quad \tau = U_1 t/a, \qquad (2.1\text{a, b})$$

and, as dependent variables, the zonal (λ) and azimuthal (φ) components of velocity, $U_1 u$ and $U_1 v$, and the potential

$$h = c_p T + g(r-a) \equiv \varkappa g H \phi(\lambda, \varphi, \sigma, \tau), \qquad (2.2)$$

where $c_p T$ is the specific enthalpy, $g(r-a)$ is the geopotential, and ϕ is dimensionless. Invoking the thermodynamic hypotheses of adiabatic motion and a perfect gas, we can derive the temperature, density and pressure from the enthalpy according to

$$T = gH/c_p)\phi_\sigma, \quad \varrho = \varrho_1 \phi_\sigma^{1(\gamma-1)} \exp[-\gamma\varkappa\sigma/(\gamma-1)],$$
$$p \hat{=} [(\gamma-1)/\gamma]\varrho_1 g H \phi_\sigma^{\gamma/(\gamma-1)} \exp[-\gamma\varkappa\sigma/(\gamma-1)], \qquad (2.3\text{a, b, c})$$

where ϱ_1 is a reference value of the density that need not be specified for the subsequent development, and γ is the usual specific-heat ratio. Substituting (2.3a) into (2.2), we obtain the dimensionless altitude

$$\zeta = (r-a)/H = \varkappa\phi - \phi_\sigma. \qquad (2.4)$$

Turning to the equations of motion, we find that the foregoing considerations, together with the restriction $H < a$ (which implies the hydrostatic approximation), allow the horizontal components of Euler's equation to be reduced to (the geostrophic approximations)

$$u = -\mathrm{Ro\,Ri}\,\phi_\varphi + O(\mathrm{Ro}),$$
$$v\cos\varphi = \mathrm{Ro\,Ri}\,\phi_\lambda + O(\mathrm{Ro}) \quad (\mathrm{Ro} \to 0), \qquad (2.5)$$

where
$$\frac{D}{D\tau} = \frac{\partial}{\partial\tau} + \frac{u}{\cos\varphi}\frac{\partial}{\partial\lambda} + v\frac{\partial}{\partial\varphi}, \qquad (2.6)$$

and subscripts denote partial differentiation. We can derive the vertical velocity $U_1 w$ from (2.4) according to

$$w = (H/a)(D\zeta/D\tau). \qquad (2.7)$$

Finally, we can place the equation of continuity in the form

$$u_\lambda + (v\cos\varphi)_\varphi + \cos\varphi(D/D\tau)\log(\varrho\zeta_\sigma) = 0. \qquad (2.8)$$

The foregoing equations of motion are essentially identical with those that formed the starting point of WELANDER's (1961) analysis, although he used pressure, rather than entropy, as an independent variable. See also MILES (1964a)

3. Eigenvalue problem

A particular solution of the foregoing equations of motion that represents a zonal wind is given by

$$\phi = \Phi(\varphi, \sigma), \quad u = U(\varphi, \sigma) = -\mathrm{Ro\,Ri}\,\Phi_\varphi,$$
$$v = w = 0, \qquad (3.1)$$

where Φ is an arbitrary (but sufficiently well behaved) function. The corresponding thermodynamic variables are given by (2.3). We consider small perturbations of this solution in the form

$$\phi = \Phi(\varphi, \sigma) + \mathcal{R}\{\psi(\varphi, \sigma)\exp[i(n\lambda - \omega t)]\}, \qquad (3.2)$$

where n is the zonal wave number, ω is the angular frequency, and $\mathcal{R}\{\}$ denotes the real part of $\{\}$. Substituting (3.2) into (2.3–(2.8), linearizing in ψ, requiring w to vanish at $\zeta = 0$, and invoking the limits

$$\mathrm{Ro} \to 0, \quad \varkappa \to 0, \qquad (3.3\text{a, b})$$

we obtain

$$(U-c)(\varrho\psi_\sigma)_\sigma + [\beta\varrho\zeta_\sigma - (\varrho U_\sigma)_\sigma]\psi = 0 \qquad (3.4)$$

and
$$U\psi_\sigma - (U-c)\psi_\sigma = 0 \quad (\zeta = 0), \qquad (3.5)$$

where $$c = (\omega a / n U_1) \cos\varphi \quad (3.6)$$

is a dimensionless wave speed, and ϱ and ζ_σ are given by (2.3b) and (2.4) with $\phi = \Phi$ therein.

We now observe that φ enters (3.4) and (3.5) only as a parameter, rather than as an independent variable; accordingly, we introduce the new independent variable

$$z = 1 - (p/p_0) = (gH/p_0) \int_0^\zeta \varrho d\zeta, \quad (3.7\text{ a, b})$$

where the subscript zero implies evaluation at $\zeta = 0$, and (3.7b) follows from (3.7a) by virtue of the hydrostatic approximation. We emphasize that $p_0 = p_0(\varphi)$. Transforming (3.4) and (3.5), we obtain

$$(U - c)(P\psi')' + [\beta - (PU')']\psi = 0 \quad (3.8)$$

and $$U'\psi - (U - c)\psi' = 0 \quad (z = 0), \quad (3.9)$$

where $$P(z) = \left(\frac{\varrho g H}{p_0}\right)^2 \zeta_\sigma = \left(\frac{\varrho g H}{p_0}\right)^2 \left(\frac{\varkappa}{Hs}\right) \quad (3.10)$$

and primes denote differentiation with respect to z with φ fixed (but we emphasize that the subscript φ in the preceding development implies partial differentiation with respect to φ with σ, not z, fixed).

To complete our statement of the eigenvalue problem, we pose the boundary condition

$$P\psi\psi' = 0 \quad (z = 1). \quad (3.11)$$

The eigenvalue problem defined by (3.8)–(3.11) is essentially identical with that posed by WELANDER (1961). PHILLIPS (1963) has stated that "theoretical treatments of [very long waves], by means of perturbation theory, are hampered by nonseparability of the linearized forms of [the equations of motion]" and implicitly criticized the analyses of Welander and others by remarking that they "assume separability". This criticism is based primarily on the fact that a solution of the eigenvalue problem defined by (3.8)–(3.11) yields ω as a (not necessarily single-valued) function of φ—viz.,

$$\omega = \omega(\varphi) = (nU_1/a)\, c[\beta(\varphi)] \sec\varphi. \quad (3.12)$$

Substituting (3.12) into (3.2) and referring to (2.5) and (3.1), we obtain the perturbation velocity

$$u - U = -\mathrm{Ro\,Ri}\,\mathcal{R}\{[\psi_\varphi - i\omega'(\varphi)\, t\psi] \exp[i(n\lambda - \omega t)]\}, \quad (3.13)$$

which cannot remain small for sufficiently large t. This implies that those eigensolutions associated with real values of c yield algebraically unstable solutions, rather than neutral modes. It remains true, however, that those eigensolutions associated with complex values of c are unstable in the conventional sense; moreover, the linear growth associated with the term $\omega'(\varphi)\, t$ is negligible compared with the exponential growth associated with the term $\exp(\omega_i t)$.

Finally, we observe that the solutions to our eigenvalue problem are not uniformly valid in the neighborhoods of either the poles or the equator. [The Rossby number defined by (1.2) is singular at the poles, whilst the coefficient $\sec\varphi$ in the equations of motion—see, e.g., (2.6)—is singular at the equator.] It appears likely that these solutions, which we may designate as geostrophic approximations, represent valid approximations in the middle latitudes to more complete solutions that are valid over the entire sphere and that, in particular, the instability predicted on the basis of the geostrophic approximation is likely to be representative of an instability that would be exhibited by the more complete solutions.

4. Domain of unstable waves

We shall devote the remaining discussion primarily to complex wave speeds for which $c_i > 0$ (but if ψ is an eigensolution for $c = c_r + ic_i$, then ψ^* is an eigensolution for $c = c^* = c_r - ic_i$ in consequence of our neglect of viscosity and heat conduction). We shall state our results without proof, regarding them as limiting cases of known results for the more general baroclinic stability problem ($\alpha, \varkappa \to 0$ in MILES 1964*a*). We use the subscripts 0 and c to denote evaluation at $z = 0$ and $z = z_c$, where $U = c$ at $z = z_c$, and introduce

$$\Lambda = \beta - (PU')'. \quad (4.1)$$

We first note the following theorems:

I. A necessary condition for instability is that Λ have the same sign as U_0' in some part of the range $0 < z < 1$.

II. Complex wave speeds for unstable waves must lie within the upper half of a circle having its center at the minimum wind speed and a radius equal to the range of wind speeds.

III. Real values of c must lie in $c < U_0$ ($U_0 > 0$) or $c < 0$ ($U_0 < 0$) if Λ and U' are positive definite in $0 \leq z \leq 1$.

IV. Complex c for which c lies in the range of U are impossible if $\Lambda_c / U_0' < 0$.

V. An upper bound to c_i is given by

$$c_i < \tfrac{1}{2} \max \left\{ (\varkappa g H \sigma_\zeta)^{-\frac{1}{2}} V U_\zeta \right\}, \qquad (4.2)$$

where V denotes the local velocity of sound.

VI. There exists one and only one unstable mode as $\beta \to 0$, with

$$c_r \to \int_0^1 U dz + O(\beta), \qquad (4.3\text{ a})$$

$$c_i \to \pi \beta K \left[\int_0^{z_c} (c - U)\, dz \right]^2 + O(\beta^2), \qquad (4.3\text{ b})$$

if and only if

$$K = -[(PU')'/P^2 U'^3]_c > 0, \qquad (4.4)$$

where z_c in (4.3b) and (4.4) denotes that point at which $U = c_r$.

This result follows from § 5 of MILES (1964*b*) on the assumption that $\alpha^2, \varkappa < \beta < 1$ therein.

VII. There exists a complex eigenvalue with the asymptotic behaviour

$$c \sim U_0 + (0.36 + 0.28 i)\, U_0'^2 \beta^{-1} + O(\beta^{-2}) \qquad (\beta \to \infty) \qquad (4.5)$$

if Λ and U' are positive definite in $0 < z < 1$.

Invoking the known theorems governing the dependence of the solutions of a linear differential equation on a parameter, we can establish that c must be a continuous function of β within the semi-circle of Theorem II provided that ψ, qua function of U, has no singularities in that semicircle. Sufficient conditions for the absence of such singularities are that P and U', qua functions of U, be analytic in this semicircle and that U' have no zeros there. These conditions are likely to be satisfied by physically admissible configurations for which Λ and U' are positive definite in $0 \leq z \leq 1$, and we then may combine Theorems II, VI, and the continuity of $c(\beta)$ to establish:

VIII. There exists one and only one unstable mode if and only if $K > 0$ and if P and U' are analytic functions of U, and have no zeros, in and on the semicircle of Theorem II.

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